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Periodic Spline Orthonormal Bases

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1. INTRODUCTION

Exponential Euler splines which have been presented in [1] constitute a system of functions whose shapes are analogous to that of the integral kernel of the Fourier transform. It is well known that the exponential Euler splines converge to the kernel when their order approaches infinity [1]. In the periodic case, it is also well known that they converge to Fourier orthonormal functions [1, 2]. But orthonormality has not been discussed in the case where the order is finite.

Legendre splines which have been presented in [3] constitute an orthonormal basis in a spline function space. But their shapes are not analogous to those of Fourier orthonormal functions.

In this paper, we shall present a system of functions which constitutes an orthonormal basis in a space of periodic spline functions and converges to the system of Fourier orthonormal functions when the order approaches infinity. In Section 2, periodic *B*-spline functions and spaces of periodic spline functions are defined and their properties are discussed as the analogy of [4]. In Section 3, we shall give an orthonormal basis in the space of periodic spline functions and show that the orthonormal basis converges to the system of Fourier orthonormal functions when the order approaches infinity.

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2. Preliminaries for the Space of Periodic Spline Functions

In this section, spaces of periodic spline functions are formulated as the analogy of [4].

DEFINITION 1. Let T, N be a real number, a natural number, respectively. Then a periodic B-spline function of order m is defined as

$$ {}_{[B]}^{m}\psi_0^N(t) \triangleq \sum_{p=-\infty}^{\infty} \{\sin(\pi p/N)/\pi p\}^m \exp(i2\pi pt/T), \qquad m=1,2,3,..., \tag{1}$$

which has period T and knots interval T/N.

The following recurrence formula is derived from (1) and the convolution theorem in the Fourier series expansion.

PROPERTY 1.

$$ {}_{[B]}^{1}\psi_0^N(t) = \begin{cases} 1, & t \in (-T/2N + qT,\ T/2N + qT), \quad q = 0, \pm 1, \pm 2, ..., \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

$$ {}_{[B]}^{m}\psi_0^N(t) = (1/T) \int_0^T {}_{[B]}^{m-1}\psi_0^N(\tau)\, {}_{[B]}^{1}\psi_0^N(t-\tau)\, d\tau, \qquad m = 2, 3, 4, ... \tag{3}$$

From (2) and (3), the periodic B-splines are expressed in the form of piecewise polynomials as follows.

PROPERTY 2.

$$ {}_{[B]}^{m}\psi_0^N(t) = mT^{1-m} \sum_{q=-\infty}^{\infty} \sum_{r=0}^{m} \{(-1)^r / r!(m-r)!\} \times \{t - ((r - m/2)/N + q)\, T\}_+^{m-1}, \qquad m = 1, 2, 3, ..., \tag{4}$$

where

$$ \{t-a\}_+^{m-1} \triangleq \begin{cases} \{t-a\}^{m-1}, & t > a, \\ 0, & t \leqslant a. \end{cases} \tag{5}$$

DEFINITION 2. We shall define the space of periodic spline functions of order m with period T and knots interval T/N as

$$ {}_{m}\vartheta_N \triangleq [{}_{[B]}^{m}\psi_l^N]_{l=0}^{N-1}, \tag{6}$$

where

$$ {}_{[B]}^{\ m}\psi_l^N(t) \triangleq {}_{[B]}^{\ m}\psi_0^N(t - lT/N), \qquad l = 0, 1, ..., N-1. \tag{7} $$

The inner product of two elements $f, g \in {}_m\mathcal{S}_N$ is defined as follows:

$$ (f, g) \triangleq (1/T) \int_0^T f(t)\,\overline{g(t)}\,dt. \tag{8} $$

As the analogy of Eq. (1.19) [1, p. 61], the inner products of periodic B-splines are obtained as follows.

PROPERTY 3.

$$ ({}_{[B]}^{\ m}\psi_{l_1}^N, {}_{[B]}^{\ m}\psi_{l_2}^N) = {}_{[B]}^{\ 2m}\psi_0^N((l_2 - l_1)\,T/N), \qquad l_1, l_2 = 0, 1, ..., N-1. \tag{9} $$

3. AN ORTHONORMAL BASIS IN THE SPACE OF PERIODIC SPLINE FUNCTIONS

In this section, we shall present an orthonormal basis in the space of periodic spline functions and show that the basis converges to the system of Fourier orthonormal functions when the order approaches infinity.

The following theorem gives an orthonormal basis in the space of periodic spline functions:

THEOREM 1. *Functions* $\{{}_{[O]}^{\ m}\psi_k^N\}_{k=0}^{N-1}$ *which are defined by*

$$ {}_{[O]}^{\ m}\psi_k^N \triangleq ({}^mH_k^H)^{-1/2}(1/N) \sum_{l=0}^{N-1} \exp(i2\pi lk/N)\, {}_{[B]}^{\ m}\psi_l^N \tag{10} $$

yield an orthogonal basis in ${}_m\mathcal{S}_N$, *where*

$$ {}^mH_k^N \triangleq \sum_{p=-\infty}^{\infty} \{\sin(\pi k/N)/\pi(k + pN)\}^{2m}, \qquad k = 0, 1, ..., N-1. \tag{11} $$

Proof. Obviously the functions $\{{}_{[O]}^{\ m}\psi_k^N\}_{k=0}^{N-1}$ yield a basis in ${}_m\mathcal{S}_N$. Let us calculate inner products of any two functions in $\{{}_{[O]}^{\ m}\psi_k^N\}_{k=0}^{N-1}$ to prove their orthonormality:

$$
\begin{aligned}
&({}_{[O]}^{m}\psi_{k_1}^{N}, {}_{[O]}^{m}\psi_{k_2}^{N}) \\
&= \left(({}^{m}H_{k_1}^{N})^{-1/2} (1/N) \sum_{l_1=0}^{N-1} \exp(i2\pi l_1 k_1/N)\, {}_{[B]}^{m}\psi_{l_1}^{N}, \right. \\
&\qquad \left. ({}^{m}H_{k_2}^{N})^{-1/2} (1/N) \sum_{l_2=0}^{N-1} \exp(i2\pi l_2 k_2/N)\, {}_{[B]}^{m}\psi_{l_2}^{N} \right) \\
&= \{N^{-2}({}^{m}H_{k_1}^{N}\, {}^{m}H_{k_2}^{N})^{-1/2}\} \\
&\quad \times \left\{ \sum_{l_1=0}^{N-1} \sum_{l_2=0}^{N-1} \exp(i2\pi(l_1k_1 - l_2k_2)/N)({}_{[B]}^{m}\psi_{l_1}^{N}, {}_{[B]}^{m}\psi_{l_2}^{N}) \right\}. \qquad (12)
\end{aligned}
$$

Making use of (9), we arrange the latter part of (12) as follows,

$$
\begin{aligned}
&\sum_{l_1=0}^{N-1} \sum_{l_2=0}^{N-1} \exp(i2\pi(l_1k_1 - l_2k_2)/N)({}_{[B]}^{m}\psi_{l_1}^{N}, {}_{[B]}^{m}\psi_{l_2}^{N}) \\
&= \sum_{l_1=0}^{N-1} \sum_{l_2=0}^{N-1} \exp(i2\pi(l_1k_1 - l_2k_2)/N)\, {}_{[B]}^{2m}\psi_{0}^{N}((l_2 - l_1)\, T/N) \\
&= \left\{ \sum_{l_1=0}^{N-1} \exp(i2\pi(k_1 - k_2)\, l_1/N) \right\} \\
&\quad \times \left\{ \sum_{l_3=0}^{N-1} \exp(-i2\pi k_2 l_3/N)\, {}_{[B]}^{2m}\psi_{0}^{N}(l_3 T/N) \right\} \qquad (l_3 \triangleq l_2 - l_1) \\
&= N\delta_{k_1-k_2} \sum_{l_3=0}^{N-1} \exp(-i2\pi k_2 l_3/N)\, {}_{[B]}^{2m}\psi_{0}^{N}(l_3 T/N), \qquad (13)
\end{aligned}
$$

where

$$
\delta_k \triangleq \begin{cases} 1 & (k=0), \\ 0 & (k \neq 0). \end{cases}
$$

Let $\delta(t)$ denote Dirac's delta function of t. Then the following equation holds good from the convolution theorem in the Fourier series expansion and (1):

$$
\begin{aligned}
&N\delta_{k_1-k_2} \sum_{l_3=0}^{N-1} \exp(-i2\pi k_2 l_3/N)\, {}_{[B]}^{2m}\psi_{0}^{N}(l_3 T/N) \\
&= N\delta_{k_1-k_2}(1/T) \int_0^T \exp(-i2\pi k_2 t/T)\, {}_{[B]}^{2m}\psi_{0}^{N}(t) \left\{ T \sum_{l_3=0}^{N-1} \delta(t - l_3 T/N) \right\} dt \\
&= N\delta_{k_1-k_2} \sum_{p=-\infty}^{\infty} \left\{ (1/T) \int_0^T \left(T \sum_{l_3=0}^{N-1} \delta(t - l_3 T/N) \right) \exp(-i2\pi pt/T)\, dt \right\} \\
&\quad \times \left\{ (1/T) \int_0^T {}_{[B]}^{2m}\psi_{0}^{N}(t) \exp(-i2\pi(k_2 - p)\, t/T)\, dt \right\}
\end{aligned}
$$

$$= N\delta_{k_1-k_2} \sum_{p=-\infty}^{\infty} \left\{ \sum_{l_3=0}^{N-1} \exp(-i2\pi p l_3/N) \right\} \{\sin(\pi(k_2-p)/N)/\pi(k_2-p)\}^{2m}$$

$$= N\delta_{k_1-k_2} \sum_{p=-\infty}^{\infty} \{N\delta_{p \bmod N}\}\{\sin(\pi(k_2-p)/N)/\pi(k_2-p)\}^{2m}$$

$$= N^2\delta_{k_1-k_2} \sum_{p=-\infty}^{\infty} \{\sin(\pi k_2/N)/\pi(k_2-pN)\}^{2m}$$

$$= N^2\delta_{k_1-k_2}\, {}^mH_{k_2}^N. \tag{14}$$

Substituting (13) and (14) for (12), we have

$$({}_{[O]}^{\,m}\psi_{k_1}^N, {}_{[O]}^{\,m}\psi_{k_2}^N) = \delta_{k_1-k_2}, \qquad k_1, k_2 = 0, 1, ..., N-1. \tag{15}$$

From the above, the functions $\{{}_{[O]}^{\,m}\psi_k^N\}_{k=0}^{N-1}$ yield an orthonormal basis in ${}_m\mathscr{S}_N$. Q.E.D.

Now we shall consider that $\{{}_{[F]}\psi_r\}_{r=-\infty}^{\infty}$ denote the system of Fourier orthonormal functions, i.e.,

$${}_{[F]}\psi_r(t) \triangleq \exp(i2\pi rt/T), \qquad r = 0, \pm 1, \pm 2, ... \tag{16}$$

Then the inner products of the orthonormal basis in a space of periodic spline functions with the Fourier are obtained as the following theorem.

THEOREM 2.

$$({}_{[O]}^{\,m}\psi_k^N, {}_{[F]}\psi_r) = ({}^mH_k^N)^{-1/2}\{\sin(\pi r/N)/\pi r\}^m\, \delta_{(k-r) \bmod N},$$

$$k = 0, 1, ..., N-1; \quad r = 0, \pm 1, \pm 2, ... \tag{17}$$

Proof. Substituting (10), (11), and (16) for (8), we have

$$({}_{[O]}^{\,m}\psi_k^N, {}_{[F]}\psi_r)$$

$$= ({}^mH_k^N)^{-1/2}(1/N) \sum_{l=0}^{N-1} \exp(i2\pi lk/N)({}_{[B]}^{\,m}\psi_l^N, {}_{[F]}\psi_r)$$

$$= ({}^mH_k^N)^{-1/2}\{\sin(\pi r/N)/\pi r\}^m(1/N) \sum_{l=0}^{N-1} \exp(i2\pi l(k-r)/N)$$

$$= ({}^mH_k^N)^{-1/2}\{\sin(\pi r/N)/\pi r\}^m\, \delta_{(k-r) \bmod N}. \qquad \text{(Q.E.D.)}$$

The following theorem shows an extremum property of the orthonormal basis in ${}_m\mathscr{S}_N$ when m approaches infinity.

THEOREM 3. *When m approaches infinity,*

$$({}_{[O]}^{m}\psi_k^N, {}_{[F]}\psi_r) \to \begin{cases} \delta_{(k-r) \bmod N}, & |r| < N/2, \\ 0, & |r| > N/2. \end{cases} \tag{18}$$

Proof. Substituting (11) for (17), we have

$$\begin{aligned} &({}_{[O]}^{m}\psi_k^N, {}_{[F]}\psi_r) \\ &= \left\{ \sum_{p=-\infty}^{\infty} \{\sin(\pi k/N)/\pi(k+pN)\}^{2m} \right\}^{-1/2} \times \{\sin(\pi r/N)/\pi r\}^m \, \delta_{(k-r) \bmod N}, \\ &\qquad k = 0, 1, ..., N-1; \quad r = k + qN. \end{aligned} \tag{19}$$

Since (19) is nonzero if and only if $(k-r) \bmod N = 0$, let us consider this case and assume $r = k + qN$ ($q = 0, \pm 1, \pm 2, ...$). Then (19) yields

$$\begin{aligned} &({}_{[O]}^{m}\psi_k^N, {}_{[F]}\psi_r) \\ &\quad = \left\{ \sum_{p=-\infty}^{\infty} ((k+qN)/(k+pN))^{2m} \right\}^{-1/2} \operatorname{sgn}[\{\sin(\pi r/N)/\pi r\}^m], \\ &\qquad k = 0, 1, ..., N-1; \quad r = k + qN \quad (q = 0, \pm 1, \pm 2, ...). \end{aligned} \tag{20}$$

We define ${}^mH_k'^N$ as

$${}^mH_k'^N \triangleq \sum_{p=-\infty}^{\infty} ((k+qN)/(k+pN))^{2m} \tag{21}$$

and study its behavior when m approaches infinity. Put $p' \triangleq p - q$. Then we have

$$\begin{aligned} {}^mH_k'^N &= \sum_{p'=-\infty}^{\infty} ((k+qN)/(k+p'N+qN))^{2m} \\ &= \sum_{p'=-\infty}^{\infty} (r/(r+p'N))^{2m} \\ &= 1 + (2r/N)^{2m} \sum_{p'=1}^{\infty} \{(2r/N+2p')^{-2m} + (2r/N-2p')^{-2m}\}. \end{aligned} \tag{22}$$

Case 1. $|r| > N/2$. Since it obviously holds good that

$$\sum_{p'=1}^{\infty} \{(2r/N+2p')^{-2m} + (2r/N-2p')^{-2m}\} \geqslant 1, \tag{23}$$

${}^{m}H_k^{\prime N}$ approaches infinity when m tends to infinity. This gives that

$$({}_{[O]}^{m}\psi_k^N, {}_{[F]}\psi_r) \to 0 \qquad (m \to \infty).$$

Case 2. $|r| < N/2$. Expression $\{(2r/N+2p')^{-2m} + (2r/N-2p')^{-2m}\}$ is a convex function of r in the domain $|r| \leqslant N/2$ and has its maximum value at $r = \pm N/2$. Then it holds good, when $|r| < N/2$, that

$$\begin{aligned}
&\sum_{p'=1}^{\infty} \{(2r/N+2p')^{-2m} + (2r/N-2p')^{-2m}\} \\
&< \sum_{p'=1}^{\infty} \{(1+2p')^{-2m} + (1-2p')^{-2m}\} \\
&= -1 + 2\sum_{p'=1}^{\infty} (2p'-1)^{-2m} \\
&\leqslant -1 + 2\sum_{p'=1}^{\infty} (2p'-1)^{-2} \\
&= (\pi^2-4)/4.
\end{aligned} \tag{24}$$

Therefore, ${}^{m}H_k^{\prime N}$ converges to 1 when m approaches infinity. Paying attention to $\{\sin(\pi r/N)/\pi r\}^m > 0$ for $|r| < N/2$, we have

$$({}_{[O]}^{m}\psi_k^N, {}_{[F]}\psi_r) \to \delta_{(k-r) \bmod N} \qquad (m \to \infty).$$

From the above, Theorem 3 holds good. Q.E.D.

COROLLARY 1. *In the case that N is odd, when m approaches infinity, it holds good that*

$${}_{[O]}^{m}\psi_k^N \to \begin{cases} {}_{[F]}\psi_k, & k = 0, 1, ..., (N-1)/2, \\ {}_{[F]}\psi_{k-N}, & k = (N+1)/2, ..., N-1. \end{cases} \tag{25}$$

4. PHYSICAL MEANING OF THE ORTHONORMAL BASIS

Corollary 1 shows that the orthonormal basis in ${}_{m}\mathcal{S}_N$ converges to the system of Fourier orthonormal functions when the order approaches infinity if the dimension N is odd. We may say that the orthonormal basis introduces some physical concept like the harmonic frequency into a space of periodic spline functions. We shall call such concept "fluency."

Convergence in the case of even dimension is left for further investigation in the future.

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